1. Artin's Theorem.

Statement :- Let G be a finite group of Aut(K) and F_0 is fixed subfield under G.

i.e. $F_0 = \{ \lambda \in K ; \sigma(\lambda) = \lambda \ \forall \ \sigma \in G \}$

T.P $[K : F_0] = O(G)$

Def : Aut(K) = A mapping $f : G \to G$

i) f is 1-1, onto & homomorphism

called Aut. on group G.

Def : fixed field :- Let G be a group of Aut. of field K.

Then subfield F_0 of K consist. of all those element $x \in K$ s.t. $\sigma(x) = x \ \forall \ \sigma \in G$ is called fixed field under G and denoted by K_G.

Proof :- Let $[K : F_0] = m$

& $O(G) = n$

T.P $m = n$

Let $\sigma_1, \sigma_2 \ldots \sigma_n$ are Element of G &

Let $\{x_1, x_2 \ldots x_m\}$ be a basis of K over F

Case I If possible suppose $m < n$

Consider a system of m linear homogeneous Eqn, $1 \le j \le m$

$$\sigma_1(x_j) u_1 + \sigma_2(x_j) u_2 + \cdots + \sigma_n(x_j) u_n = 0 \quad (1)$$

$\sigma_1(x_j), \sigma_2(x_j) \cdots \sigma_n(x_j)$ are Element of K &

$u_1\ u_2 \cdots u_n$ are variables.

Eq (1) has non-trivial soln.

say $y_1\ y_2 \cdots y_n$ not all y_i's are zero.

$$\sigma_1(x_j)y_1 + \sigma_2(x_j)y_2 + \cdots + \sigma_n(x_j)y_n = 0 \quad — (2)$$

if $x \in K$ then

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_m x_m \quad , \alpha_i \in F_0$$

multiply (2) by α_j,

$$\sigma_1(x_j)y_1\alpha_j + \sigma_2(x_j)y_2\alpha_j + \cdots + \sigma_n(x_j)y_n\alpha_j = 0$$

$$\Rightarrow \sigma_1(x_j)\ \sigma_1(\alpha_j)y_1 + \sigma_2(x_j)\ \sigma_2(\alpha_j)y_2 + \cdots + \sigma_n(x_j)\sigma_n(\alpha_j)y_n = 0$$

$$\Rightarrow \sigma_1(x_j\alpha_j)y_1 + \sigma_2(x_j\alpha_j)y_2 + \cdots + \sigma_n(x_j\alpha_j)y_n = 0$$

Now $\sigma_1(\alpha_1 x_1)y_1 + \sigma_2(\alpha_1 x_1)y_2 + \cdots + \sigma_n(x_1\alpha_1)y_n = 0$ — (a)

$\sigma_1(\alpha_2 x_2)y_1 + \sigma_2(\alpha_2 x_2)y_2 + \cdots \sigma_n(x_2\alpha_2)y_n = 0$ — (b)

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$\sigma_1(\alpha_m x_m)y_1 + \sigma_2(\alpha_m x_m)y_2 + \cdots \sigma_n(x_m\alpha_m)y_n = 0$ — (c)

Add All, we get

$$\sigma_1(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m)y_1 + \sigma_2(\alpha_1 x_1 + \alpha_2 x_2 + \cdots \alpha_m x_m)y_2$$
$$+ \cdots + \sigma_n(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m)y_n = 0$$

$$\Rightarrow \sigma_1(x)y_1 + \sigma_2(x)y_2 + \cdots + \sigma_n(x)y_n = 0 \quad \forall x \in K$$

$$(y_1\sigma_1 + y_2\sigma_2 + \cdots + y_n\sigma_n)x = 0$$

$$\Rightarrow y_1\sigma_1 + y_2\sigma_2 + \cdots + y_n\sigma_n = 0$$

atleast one of $y_j \neq 0$

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$\sigma_1, \sigma_2, \ldots, \sigma_n$ are linear Dependent over k.

contradiction as Elements of Aut k are $L.I.$

$\therefore m \not> n$.

Case II. If Possible suppose $m > n$

$\exists$ $(n+1)$ $L.I.$ Elements.

say $x_1, x_2, \ldots x_{n+1}$ in k over F_0.

Consider system of n linear homogeneous in $(n+1)$ variables.

$$\sigma_j(x_1)u_1 + \sigma_j(x_2)u_2 + \ldots + \sigma_j(x_{n+1})u_{n+1} = 0 \quad -(3)$$

homogeneous Eqn have non-Trivial soln.

Let $z_1, z_2, \ldots, z_{n+1}$ be non-Trivial soln of (3)

Let r be smallest non-zero integer s.t. $z_j = 0 \ \forall \ j \geq r+1$

Eq (3) reduce to

$$\sigma_j(x_1)z_1 + \sigma_j(x_2)z_2 + \ldots + \sigma_j(x_r)z_r = 0$$

$z_r \neq 0$ & $z_r \in k$ $\quad -(4)$

$\therefore$ Consider $z_i' = z_i/z_r$

By Eqn (4)

$$\sigma_j(x_1)z_1' + \sigma_j(x_2)z_2' + \ldots + \sigma_j(x_r) = 0 \quad -(5)$$

for $j=1$, $\sigma_1 = 1$

By (5)

$$x_1 z_1' + x_2 z_2' + \dots + x_{r-1} z'_{r-1} + x_r = 0 \quad -(6)$$

all $z_1', z_2' \dots z'_{r-1}$ are in F_0.

By (6)

$x_1 x_2 \dots x_r$ are L.D over F_0

not Possible

$\therefore x_1 x_2 \dots x_{n-1}$ are L.I.

Hence atleast one z_1' is not in F_0

say $z_i' \notin F_0$

Let $z_1' = 1$

then $z_1' \in F_0$ as $1 \in F_0$

Since $z_1' \notin F_0$

$\exists\ \sigma_i \in G$

$\sigma_i(z_1') \neq z_1$

apply σ_i to Eqn (B)

$$\sigma_i(\sigma_j(x_1) z_1') + \sigma_i(\sigma_j(x_2) z_2') + \dots +$$

$$\sigma_i(\sigma_j(x_{r-1}) z'_{r-1}) + \sigma_i(\sigma_j(x_r)) = 0$$

$$\sigma_i(\sigma_j(x_i))\sigma_i(z_1') + \sigma_i(\sigma_j(x_2))\sigma_i(z_2') + \dots$$

$$\sigma_i(\sigma_j(x_{r-1}))\sigma_i(z'_{r-1}) + \sigma_i(\sigma_j(x_r)) = 0$$

Since G is a Group

the set $\{\sigma_i \sigma_1, \sigma_i \sigma_2 \dots \sigma_i \sigma_n\}$ coincide with set $\{\sigma_1 \sigma_2 \dots \sigma_n\}$ through order of element will be diff, we get

$$\sigma_j(x_1)\sigma_i(z_1') + \sigma_j(x_2)\sigma_i(z_2') + \dots \sigma_j(x_r) = 0 \quad \text{— (7)}$$

Substract (7) from (6)

$$\sigma_j(x_1)(z_1' - \sigma_i(z_1')) + \sigma_j(x_2)(z_2' - \sigma_i(z_2')) + \dots + \sigma_j(x_{r-1})(z_{r-1}' - \sigma_i(z_{r-1}')) = 0$$

Put $t_x = z_x' - \sigma_i(z_x')$

then

$$\sigma_j(x_1)t_1 + \sigma_j(x_2)t_2 + \dots + \sigma_j(x_{r-1})t_{r-1} = 0$$

Here $t \neq 0$

Thus $\{t_1, t_2, \dots t_{r-1}, 0, 0, \dots, 0\}$ is a non trivial soln. which is contradiction to choice of r.

$\therefore m \neq n$

Hence $m = n$

$\therefore [K : F_0] = O(G)$.